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Engineering Notes

Effect of J_2 Perturbations on Relative Spacecraft Position in Near-Circular Orbits

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I. Introduction

FORMATION flying, in which several spacecraft maintain their relative positions with each other, has attracted much attention in recent years, and various missions for Earth and astronomical observations using spacecraft flying in formation have been proposed [1,2]. A spacecraft that moves in a reference orbit is called the chief spacecraft, and a spacecraft flying in formation with the chief spacecraft is called the deputy spacecraft. The relative equations of motion for a circular reference orbit are given by the well-known Hill–Clohessy–Wiltshire (HCW) equations [3]. The HCW equations are a set of linearized equations that describe the motion of the deputy relative to the chief under the assumption that the Earth is perfectly spherical. Because the effects of the Earth's oblateness (as described by the J_2 term in the Earth's gravitational potential) are ignored in the derivation of the HCW equations, they cannot be used to accurately predict relative motion over a long time period. Many studies have incorporated the effects of the J_2 perturbation. Schweighart and Sedwick extended the HCW equations to include the effect of J_2 and derived a set of constant-coefficient linearized equations of motion [4,5]. Ross [6] and Roberts and Roberts [7] derived a set of time-varying coefficient linearized equations of motion using the gradient of acceleration due to J_2.

Even when the chief orbit is nominally circular, a small eccentricity usually occurs because of J_2. In general, the relative equations of motion for a chief with an elliptic orbit are known as the Tschauner–Hempel equations [8]. The relative motion in an elliptic orbit becomes much more complicated because of J_2. Gim and Alfriend derived a state transition matrix for the relative motion under such conditions with the effects of J_2 [9]. Hamel and Lafontaine also derived a state transition matrix for the relative motion. They approximated the time variation in the difference between the osculating orbit elements of the chief and deputy by using the difference between the mean orbit elements and obtained a simple form of the matrix [10]. In addition, Humi and Carter focused on cases in which the chief follows an equatorial orbit and a polar orbit and derived relative equations of motion under J_2 [11]. An averaged solution of the relative motion when the chief is in an elliptical orbit with the J_2 perturbation was also obtained [12]. Moreover, a method to modify the initial conditions of a formation through numerical integration of the equations of motion by using the Gaussian least-square method was proposed [13].

In this study, we use the solutions to the HCW equations to focus on two formations in which the relative distance between the chief and deputy is constant: an along-track formation and a circular formation. We analyze the relative distance variation due to J_2. This distance variation is particularly crucial when formation flying is used to create a telescope with a long focal distance. In relation to the distance variation caused by J_2, Sabatini et al. used a genetic algorithm and examined the conditions required to minimize the deviation from a circular formation; they found that the chief orbit had two special inclinations at which the deviation becomes extremely small [14]. Moreover, the condition for the agreement of in-plane and out-of-plane fundamental frequencies of the relative motion under the effects of J_2 is reduced to the same inclinations as that of the chief orbit [15].

The main objective of this study is to analyze the relative distance variation within one orbit period for a mission in which the relative distance is precisely maintained. For this purpose, we take a novel approach to derive a state transition matrix without using the mean orbit elements of the chief, assuming that the eccentricity of the chief is small, i.e., of the same order of magnitude as J_2. From this state transition matrix, we obtain the relative distance variation occurring in one orbit period in an analytical form. Furthermore, we also derive the relation between the initial values of the deputy and the relative distance variation. The new formulation of the deputy initial values in terms of the osculating orbit elements is derived to reduce the distance variation. Finally, numerical simulations are carried out to validate the analytical results.

II. State Transition Matrix

A. LVLH Coordinates

A local orthogonal coordinate system with its origin at the center of mass of the chief is selected such that the x axis is aligned along an imaginary line drawn from the Earth's center of mass to the chief; the z axis is aligned along the direction normal to the orbital plane of the chief, and the y axis constitutes the right-hand system along with the x and z axis. These local coordinates are called local-vertical–local-horizontal (LVLH) coordinates. It is assumed that the J_2 forces are exerted on both the chief and the deputy in addition to the nominal gravitational forces, which are inversely proportional to the square of the distance from the center of the Earth to each spacecraft. Let the J_2 gravitational forces that are exerted on the chief per unit mass be $[f_x \quad f_y \quad f_z]$ in the LVLH coordinates. These forces are expressed using the orbit elements of the chief as follows [16]:

$$f_x = \frac{3\mu J_2 R^2}{2r_c^4}(3\sin^2 i_c \sin^2\theta_c - 1) \tag{1}$$

$$f_y = -\frac{3\mu J_2 R^2}{2r_c^4}\sin^2 i_c \sin 2\theta_c \tag{2}$$

$$f_z = -\frac{3\mu J_2 R^2}{2r_c^4}\sin 2i_c \sin\theta_c \tag{3}$$

where $J_2 = 1.0826 \times 10^{-3}$, μ is the gravity constant of the Earth, R is the equatorial radius of the Earth, r_c is the distance from the center of the Earth to the chief, i_c is the inclination of the chief orbit, θ_c is the

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true latitude of the chief, and the subscript c represents variables of the chief.

B. Transformation of State Vector

The spacecraft position vector $\mathbf{r}$ from the center of the Earth expressed in the LVLH coordinates is defined as

$$\mathbf{r} = [r_x \quad r_y \quad r_z]^T \tag{4}$$

and the spacecraft state $\mathbf{p}$ is defined as

$$\mathbf{p} = [r_x \quad r_y \quad r_z \quad \dot{r}_x \quad \dot{r}_y \quad \dot{r}_z]^T \tag{5}$$

The relative state of the deputy with respect to that of the chief is expressed as $\delta\mathbf{p} = \mathbf{p}_d - \mathbf{p}_c$, where the subscript d represents variables of the deputy. The spacecraft orbit elements are defined as

$$\mathbf{s} = [a \quad e \quad i \quad \Omega \quad \omega \quad M]^T \tag{6}$$

where a is the semimajor axis, e is the eccentricity, Ω is the right ascension of the ascending node, ω is the argument of the perigee, and M is the mean anomaly. The deviation of the orbit elements of the deputy from those of the chief are expressed as $\delta\mathbf{s} = \mathbf{s}_d - \mathbf{s}_c$. We now compute the time evolution of $\delta\mathbf{s}$ and derive the relationship between $\delta\mathbf{p}(\theta_c)$ and $\delta\mathbf{p}(0)$, where $\delta\mathbf{p}$ is regarded as a function of the true latitude θ_c. It is assumed that in the nominal state, the deputy is located at a certain distance from the chief, the ratio of this distance to the semimajor axis of the chief a_c is small, and the eccentricity of the chief is also small. Under these assumptions, the relationships required for calculating the state transition matrix in the LVLH coordinates are derived by neglecting the second- and higher-order terms of J_2 and the aforementioned small quantities.

C. Transformation Between $\delta\mathbf{p}$ and $\delta\mathbf{s}$

The transformation between $\delta\mathbf{p}$ and $\delta\mathbf{s}$ is expressed as

$$\delta\mathbf{p} = \mathbf{A}_{Hs}\delta\mathbf{s}, \qquad \mathbf{A}_{Hs} = \left.\frac{\partial\mathbf{p}}{\partial\mathbf{s}}\right|_c \tag{7}$$

$$\delta\mathbf{s} = \mathbf{A}_{sH}\delta\mathbf{p}, \qquad \mathbf{A}_{sH} = \mathbf{A}_{Hs}^{-1} = \left.\frac{\partial\mathbf{s}}{\partial\mathbf{p}}\right|_c \tag{8}$$

where the subscript c of the partial derivatives signifies that the evaluation is made at the state (orbit elements) of the chief. The matrices $\mathbf{A}_{Hs}$ and $\mathbf{A}_{sH}$ are easily derived from [9].

D. Time Evolution of $\delta\mathbf{s}$

Next, we compute the time evolution of $\delta\mathbf{s}$. By determining the time evolution, we can obtain the state transition matrix in the LVLH coordinates by multiplying the above coefficient matrices. Gauss's variational equations expressing the variations of the osculating orbit elements are expressed as follows [17]:

$$\frac{\mathrm{d}a_c}{\mathrm{d}t} = \frac{2a_c^2}{h_c}\left(e_c \sin\nu_c f_x + \frac{p_c}{r_c} f_y\right) \tag{9}$$

$$\frac{\mathrm{d}e_c}{\mathrm{d}t} = \frac{1}{h_c}[p_c \sin\nu_c f_x + ((p_c + r_c)\cos\nu_c + r_c e_c) f_y] \tag{10}$$

$$\frac{\mathrm{d}i_c}{\mathrm{d}t} = \frac{r_c \cos\theta_c}{h_c} f_z \tag{11}$$

$$\frac{\mathrm{d}\Omega_c}{\mathrm{d}t} = \frac{r_c \sin\theta_c}{h_c \sin i_c} f_z \tag{12}$$

$$\frac{\mathrm{d}\omega_c}{\mathrm{d}t} = \frac{1}{h_c e_c}[-p_c \cos\nu_c f_x + (p_c + r_c)\sin\nu_c f_y] - \frac{r_c \sin\theta_c \cos i_c}{h_c \sin i_c} f_z \tag{13}$$

$$\frac{\mathrm{d}M_c}{\mathrm{d}t} = n_c + \frac{b_c}{a_c h_c e_c}[(p_c \cos\nu_c - 2r_c e_c) f_x - (p_c + r_c)\sin\nu_c f_y] \tag{14}$$

where ν_c is the true anomaly of the chief, n_c is the mean angular motion of the chief, $p_c = a_c(1 - e_c^2)$, $h_c = n_c\sqrt{1 - e_c^2}a_c^2$, and $b_c = a_c\sqrt{1 - e_c^2}$. Gauss's variational equations can be expressed as

$$\frac{\mathrm{d}\mathbf{s}}{\mathrm{d}t} = \mathbf{w} \tag{15}$$

From this equation, the time evolution of $\delta\mathbf{s}$ is obtained as

$$\frac{\mathrm{d}\delta\mathbf{s}}{\mathrm{d}t} = \mathbf{A}_{\delta s}\delta\mathbf{s}, \qquad \mathbf{A}_{\delta s} = \frac{\partial\mathbf{w}}{\partial\mathbf{s}} \tag{16}$$

Because the exact solution of this equation cannot be analytically obtained, the perturbed solution is expressed by considering the J_2 term as small. First, $\delta\mathbf{s}$ is regarded as a function of time t, where $t = 0$ is defined as the time at the ascending node $\theta_c = 0$ of the chief. $\mathbf{A}_{\delta s}$ and $\delta\mathbf{s}$ can be expressed as the sum of the terms that depend on J_2 (subscript J) and those that are independent of J_2 (subscript N), respectively, as

$$\mathbf{A}_{\delta s} = \mathbf{A}_{\delta sN} + \mathbf{A}_{\delta sJ} \tag{17}$$

$$\delta\mathbf{s} = \delta\mathbf{s}_N + \delta\mathbf{s}_J \tag{18}$$

From these equations, the following equations are obtained as the zeroth- and first-order differential equations:

$$\frac{\mathrm{d}\delta\mathbf{s}_N}{\mathrm{d}t} = \mathbf{A}_{\delta sN}\delta\mathbf{s}_N \tag{19}$$

$$\frac{\mathrm{d}\delta\mathbf{s}_J}{\mathrm{d}t} = \mathbf{A}_{\delta sN}\delta\mathbf{s}_J + \mathbf{A}_{\delta sJ}\delta\mathbf{s}_N \tag{20}$$

As for the matrix $\mathbf{A}_{\delta sN}$, the (6,1) component has the value $-3n_c/(2a_c)$, but all other components are 0. In addition, because the product of δa and $-3n_c/(2a_c)$ is also small, the above equations are further simplified by substituting the initial values $\delta\mathbf{s}_N(0)$ and $\delta\mathbf{s}_J(0)$ into $\delta\mathbf{s}_N$ and $\delta\mathbf{s}_J$ on the right-hand side, respectively. Then, $\delta\mathbf{s}_N$ and $\delta\mathbf{s}_J$ are given by

$$\delta\mathbf{s}_N(t) = (\mathbf{I}_6 + \mathbf{A}_{\delta sN}t)\delta\mathbf{s}_N(0) \tag{21}$$

$$\delta\mathbf{s}_J(t) = (\mathbf{I}_6 + \mathbf{A}_{\delta sN}t)\delta\mathbf{s}_J(0) + \int_0^t \mathbf{A}_{\delta sJ}\delta\mathbf{s}_N(t)\,\mathrm{d}t \tag{22}$$

where $\mathbf{I}_6$ is a 6×6 unit matrix. Thus, we can obtain $\delta\mathbf{s}(t)$ as $\delta\mathbf{s}_N(t) + \delta\mathbf{s}_J(t)$. In the calculation of Eqs. (21) and (22), the integration over t is transformed into that over θ_c by using the relation $h_c = r_c^2\dot{\theta}_c$. When time t is explicitly included in the integrated terms or the integration cannot be executed analytically, r_c is expanded around $e_c = 0$ as

$$r_c = a_c(1 - \cos\nu_c e_c - \sin^2\nu_c e_c^2 + \cdots) \tag{23}$$

From this equation, time t can also be approximated by the true anomaly ν_c as

$$t = \frac{\nu_c + \omega_c}{n_c} - \frac{2(\sin\nu_c + \sin\omega_c)}{n_c} e_c + \frac{3(\sin\nu_c \cos\nu_c + \sin\omega_c \cos\omega_c)}{2n_c} e_c^2 + \cdots \tag{24}$$

By carrying out these calculations, $\delta\mathbf{s}(\mathbf{t})$ can be changed into a function of the true latitude θ_c, $\delta\mathbf{s}(\theta_c)$.

E. Time Evolution of $\delta\mathbf{p}$

When $\delta\mathbf{s}(\theta_c)$ is calculated, $\delta\mathbf{p}(\theta_c)$ is obtained by multiplying $\mathbf{A}_{Hs}$. As $\mathbf{A}_{Hs}$ becomes a function of the true latitude θ_c, this matrix is expressed as $\mathbf{A}_{Hs}(\theta_c)$. Then, the following equation is obtained:

$$\delta\mathbf{p}(\theta_c) = \mathbf{A}_{Hs}(\theta_c)\delta\mathbf{s}(\theta_c) \tag{25}$$

In this equation, the terms in $\mathbf{A}_{Hs}(\theta_c)$ are composed of the orbit elements $(a_c, e_c, i_c, \Omega_c, \omega_c, \theta_c)$, and these elements vary due to J_2. In the calculation of $\mathbf{A}_{Hs}(\theta_c)$, the variation in each orbit element needs to be considered. Let each orbit element be u_i $(i = 1, 2, \cdots, 6)$, and let $\mathbf{A}_{Hs}(\theta_c)$ be $\mathbf{A}_{HsN}(\theta_c)$ when the variation in u_i is not considered. Then, $\mathbf{A}_{Hs}(\theta_c)$ can be calculated from $\mathbf{A}_{HsN}(\theta_c)$ as

$$\mathbf{A}_{Hs}(\theta_c) = \mathbf{A}_{HsN}(\theta_c) + \sum_{i=1}^{6} \frac{\partial \mathbf{A}_{Hs}}{\partial u_i}\bigg|_N \Delta u_i \tag{26}$$

where $\partial\mathbf{A}_{Hs}/\partial u_i|_N$ denotes $\partial\mathbf{A}_{Hs}/\partial u_i$ without the effects of J_2, and Δu_i denotes the variation in the orbit element u_i caused by J_2. Δu_i is obtained as

$$\Delta u_i = \int_0^{\theta_c} \frac{\mathrm{d}u_i}{\mathrm{d}t}\frac{r_c^2}{h_c}\,\mathrm{d}\theta_c \tag{27}$$

$\mathrm{d}u_i/\mathrm{d}t$ in this equation can be calculated using Eqs. (9–14). Special attention should be paid to the calculation of $\Delta\theta_c$, because the nominal orbit rate is included in the time derivative of θ_c. First, on the basis of the relations

$$\tan\frac{\nu_c}{2} = \sqrt{\frac{1+e_c}{1-e_c}}\tan\frac{E_c}{2} \tag{28}$$

$$E_c - e_c \sin E_c = M_c \tag{29}$$

where E_c is the eccentric anomaly of the chief, we obtain the time derivative of $\theta_c = \nu_c + \omega_c$ as

$$\frac{\mathrm{d}\theta_c}{\mathrm{d}t} = \frac{(1+e_c\cos\nu_c)^2}{(1-e_c^2)^{\frac{3}{2}}}\frac{\mathrm{d}M_c}{\mathrm{d}t} + \frac{\mathrm{d}\omega_c}{\mathrm{d}t} + \frac{\sin\nu_c(2+e_c\cos\nu_c)}{1-e_c^2}\frac{\mathrm{d}e_c}{\mathrm{d}t} \tag{30}$$

The aforementioned equation can be expressed in terms of v_{cN}, which is independent of J_2, and v_{cJ}, which depends on J_2, as

$$\frac{\mathrm{d}\theta_c}{\mathrm{d}t} = v_{cN} + v_{cJ}, \qquad v_{cN} = \frac{(1+e_{c0}\cos\nu_c)^2 n_{c0}}{(1-e_{c0}^2)^{\frac{3}{2}}} \tag{31}$$

where subscript 0 denotes the osculating orbit elements at the ascending node $(\theta_c = 0)$ of the chief. Then, $\Delta\theta_c$ can be expressed as

$$\Delta\theta_c = \int_0^{\theta_c} \frac{\mathrm{d}v_{cN}}{\mathrm{d}\theta_c}\Delta\theta_c\frac{r_c^2}{h_c}\,\mathrm{d}\theta_c + \int_0^{\theta_c} v_{cJ}\frac{r_c^2}{h_c}\,\mathrm{d}\theta_c \tag{32}$$

By differentiating both sides of this equation with respect to θ_c and rearranging the terms, we obtain the following equation for $\Delta\theta_c$:

$$\frac{\mathrm{d}\Delta\theta_c}{\mathrm{d}\theta_c} = -\frac{2e_{c0}\sin\nu_c}{1+e_{c0}\cos\nu_c}\Delta\theta_c + \frac{v_{cJ}(1-e_{c0}^2)^{\frac{3}{2}}}{n_{c0}(1+e_{c0}\cos\nu_c)^2} \tag{33}$$

By considering only the first-order terms of J_2, $\Delta\theta_c$ can be obtained as the solution of this differential equation as

$$\Delta\theta_c = (1+e_{c0}\cos\nu_c)^2\int_0^{\theta_c}\frac{v_{cJ}(1-e_{c0}^2)^{\frac{3}{2}}}{n_{c0}(1+e_{c0}\cos\nu_c)^4}\,\mathrm{d}\theta_c \tag{34}$$

All the components of Δu_i can be obtained from Eqs. (27) and (34).

By using the above equations, $\delta\mathbf{p}(\theta_c)$ can be calculated from $\delta\mathbf{p}(0)$ as follows: $\delta\mathbf{s}(0)$ is obtained from $\mathbf{A}_{sH}$ as

$$\delta\mathbf{s}(0) = \mathbf{A}_{sH}(0)\delta\mathbf{p}(0) \tag{35}$$

where $\mathbf{A}_{sH}$ is regarded as a function of the true latitude θ_c. Then, $\delta\mathbf{s}(\theta_c)$ is obtained from $\delta\mathbf{s}(0)$. By using Eqs. (25–27), $\delta\mathbf{p}(\theta_c)$ is obtained from $\delta\mathbf{s}(\theta_c)$. The state transition matrix $\mathbf{\Phi}(\theta_c, 0)$ that translates $\delta\mathbf{p}(0)$ to $\delta\mathbf{p}(\theta_c)$ is obtained as

$$\mathbf{\Phi}(\theta_c, 0) = \mathbf{A}_{Hs}(\theta_c)\mathbf{A}_{\delta s}(\theta_c, 0)\mathbf{A}_{sH}(0) \tag{36}$$

III. Relative Distance Variation During Orbit Period

By employing the state transition matrix $\mathbf{\Phi}(\theta_c, 0)$, we can evaluate the distance variation between the chief and the deputy during one orbit period. Here, the focus is on the following two cases: i) an along-track formation wherein the chief is in a circular orbit and the deputy moves in the same orbit as that of the chief and at a constant distance from the chief, and ii) a circular formation wherein the chief is in a circular orbit and the deputy makes a circular formation with respect to the chief. How the relative distances are affected by J_2 is analyzed in the following sections.

A. Along-Track Formation

It is assumed that the initial values of the deputy at the chief ascending node are given by the following $\delta\mathbf{p}_0$:

$$\delta\mathbf{p}_0 = [0 \quad d_a \quad 0 \quad 0 \quad 0 \quad 0]^T \tag{37}$$

where d_a is the distance in the moving direction. According to the procedure in the previous section, the relative position and velocity of the deputy at the true latitude θ_c are obtained from $\delta\mathbf{p}(0)$. The components of the relative position and velocity $\delta\mathbf{p}(\theta_c)$ are assumed to be

$$\delta\mathbf{p}(\theta_c) = [p_x \quad p_y \quad p_z \quad v_x \quad v_y \quad v_z]^T \tag{38}$$

The relative distance variation Δd is given by

$$\Delta d = \sqrt{p_x^2 + p_y^2 + p_z^2} - d_a \tag{39}$$

By substituting $\delta\mathbf{p}(0) = \delta\mathbf{p}_0$ and using $\mathbf{\Phi}(\theta_c, 0)$ in Eq. (36), the following Δd_a is obtained as

$$\begin{aligned}\Delta d_a = &-\frac{\alpha d_a}{2}[\sin^2 i_{c0}(1-\cos 2\theta_c) + (5 - 2\cos^2 i_{c0})(1-\cos\theta_c)]\\ &+ e_{c0}\mathrm{d}_a(-3\theta_c\sin\omega_{c0} + 3\sin\omega_{c0}\sin\theta_c - \cos\omega_{c0}\cos\theta_c\\ &+ \cos\omega_{c0})\end{aligned} \tag{40}$$

where α is set as

$$\alpha = \frac{J_2 R^2}{a_{c0}^2}$$

As is clear from Eq. (40), the relative distance varies with the period of the chief and with twice the period of the chief. To reduce this distance variation by properly setting the initial values of the deputy at the ascending node of the chief, we first change the initial values of the deputy by a small quantity $\Delta\delta\mathbf{p}_0$ from those given by $\delta\mathbf{p}_0$ in (37) as

$$\Delta\delta\mathbf{p}_0 = [\,\Delta p_{x0} \quad \Delta p_{y0} \quad \Delta p_{z0} \quad \Delta v_{x0} \quad \Delta v_{y0} \quad \Delta v_{z0}\,]^T \tag{41}$$

where Δp_{x0}, Δp_{y0}, and Δp_{z0} are assumed to be of the order of $J_2 d_a$; and Δv_{x0}, Δv_{y0}, and Δv_{z0} are assumed to be of the order of $J_2 n_{c0} d_a$. The relative distance Δd is set as follows when the relative initial values of the deputy are $\delta\mathbf{p}(0) = \delta\mathbf{p}_0 + \Delta\delta\mathbf{p}_0$:

$$\Delta d = \Delta d_a + \epsilon \tag{42}$$

ϵ appearing in this equation is given as follows (the second- and higher-order terms of J_2 are neglected):

$$\epsilon = -6(\theta_c - \sin\theta_c)\Delta p_{x0} + \Delta p_{y0} - 2(1-\cos\theta_c)\frac{\Delta v_{x0}}{n_{c0}} - (3\theta_c - 4\sin\theta_c)\frac{\Delta v_{y0}}{n_{c0}} \tag{43}$$

As is clear from Eqs. (40) and (43), the term with the same period as that of the chief can be eliminated by setting Δp_{x0} and Δv_{x0} as

$$\Delta p_{x0} = -\frac{1}{2} d_a e_{c0} \sin\omega_{c0} \tag{44}$$

$$\Delta v_{x0} = \frac{n_{c0} d_a}{4}[\alpha(2\cos^2 i_{c0} - 5) + 2e_{c0}\cos\omega_{c0}] \tag{45}$$

When Δp_{x0} and Δv_{x0} are set as Eq. (44) and (45), respectively, and all other components of $\Delta\delta\mathbf{p}_0$ are 0, Δd in Eq. (42) becomes

$$\Delta d = -\frac{\alpha d_a \sin^2 i_{c0}(1-\cos 2\theta_c)}{2} \tag{46}$$

Because Δd in Eq. (46) is only the first term of Δd_a in Eq. (40), it is clear that the relative distance variation is largely decreased when the initial values of the deputy are fixed according to Eqs. (44) and (45).

B. Circular Formation

The case of a general circular formation (GCO) [18] is considered. The initial values of the deputy are set as follows:

$$\delta\mathbf{p}_0 = \begin{bmatrix}(d_c\sin\phi)/2 & d_c\cos\phi & (\sqrt{3}d_c\sin\phi)/2 \\ (n_{c0}d_c\cos\phi)/2 & -n_{c0}d_c\sin\phi & (\sqrt{3}n_{c0}d_c\cos\phi)/2\end{bmatrix}^T \tag{47}$$

where d_c is the nominal value of the formation radius and ϕ is an arbitrary phase angle. The solution of the HCW equation, without considering the J_2 term, is given as follows:

$$p_x = \frac{d_c}{2}\sin(n_{c0}t + \phi), \qquad p_y = d_c\cos(n_{c0}t+\phi)$$
$$p_z = \frac{\sqrt{3}d_c}{2}\sin(n_{c0}t+\phi) \tag{48}$$

Then, the deviation in the radius with J_2 is expressed in the following form:

$$\Delta d = \sqrt{p_x^2 + p_y^2 + p_z^2} - d_c \tag{49}$$

The result of the expansion of Eq. (49) in the first order of J_2 with the initial value of $\delta\mathbf{p}(0) = \delta\mathbf{p}_0$ is set as Δd_c. The concrete form of Δd_c is obtained by the state transition matrix $\mathbf{\Phi}(\theta_c, 0)$ and is given in the appendix.

As shown in the appendix, there are some secular terms that include the true latitude θ_c directly, in addition to the periodic terms composed of $\sin\theta_c$, $\sin 2\theta_c$, etc. When the secular terms exist, the relative distance gradually diverges. Therefore, these terms should be eliminated. Here, we consider eliminating these terms by suitably setting the initial values of the deputy and the orbit elements of the chief.

As with Eq. (41), we change the initial values of the deputy at the chief ascending node by a small quantity $\Delta\delta\mathbf{p}_0$ from those given by $\delta\mathbf{p}_0$ in Eq. (47). Then, Δd of Eq. (49) is given by

$$\Delta d = \Delta d_c + \epsilon \tag{50}$$

Here, the value ϵ is given as follows (the second- and higher-order terms of J_2 and e_{c0} are neglected):

$$\begin{aligned}\epsilon = {} & \left[-6\theta_c\cos(\theta_c+\phi) + \frac{9}{4}\sin(2\theta_c+\phi) + 2\sin(\theta_c+\phi) - \frac{15}{4}\sin\phi\right]\Delta p_{x0} + \cos(\theta_c+\phi)\Delta p_{y0} + \frac{\sqrt{3}}{4}[\sin(2\theta_c+\phi) \\ & + \sin\phi]\Delta p_{z0} + \left[\frac{3}{4}\cos(2\theta_c+\phi) - 2\cos(\theta_c+\phi) + \frac{5}{4}\cos\phi\right]\frac{\Delta v_{x0}}{n_{c0}} + \left[-3\theta_c\cos(\theta_c+\phi) + \frac{3}{2}\sin(2\theta_c+\phi) \right. \\ & \left. + \sin(\theta_c+\phi) - \frac{5}{2}\sin\phi\right]\frac{\Delta v_{y0}}{n_{c0}} + \frac{\sqrt{3}}{4}[-\cos(2\theta_c+\phi) + \cos\phi]\frac{\Delta v_{z0}}{n_{c0}}\end{aligned} \tag{51}$$

Because Δp_{x0} and Δv_{y0} cause secular terms in this equation, these two components can eliminate the secular terms proportional to $\theta_c\sin\theta_c$ and $\theta_c\cos\theta_c$ in Δd_c. For example, when Δv_{y0} is used to eliminate the secular terms, Δp_{x0} and Δv_{y0} are given by

$$\Delta p_{x0} = 0 \tag{52}$$

$$\Delta v_{y0} = -\frac{n_{c0}d_c}{4}[\alpha(2\sqrt{3}c\phi\sin 2i_{c0} + 3s\phi) + 2e_{c0}(\sin\omega_{c0}\cos\phi + 3\cos\omega_{c0}\sin\phi)] \tag{53}$$

where $s\phi = \sin\phi$ and $c\phi = \cos\phi$. The secular terms proportional to $\theta_c\sin\theta_c$ and $\theta_c\cos\theta_c$ in Δd_c are eliminated by Eqs. (52) and (53). The secular terms with respect to θ_c in Δd_c appear in proportion to $\sin^2 i_{c0}\sin 2\phi$. Thus, to eliminate these terms, it is appropriate to set the initial phase angle ϕ to satisfy the following equation:

$$\sin 2\phi = 0 \tag{54}$$

It follows that $s\phi = 0$ or $c\phi = 0$. In each case, the secular terms that depend on $\theta_c\sin 2\theta_c$ in Δd_c have the following values:

$$s\phi = 0\text{: } -\frac{9}{32}\alpha d_c(3 - 7\cos^2 i_{c0})\theta_c\sin 2\theta_c$$
$$c\phi = 0\text{: } \frac{9}{32}\alpha d_c(1 - 5\cos^2 i_{c0})\theta_c\sin 2\theta_c$$

Therefore, to eliminate these terms, the inclination of the chief i_{c0} should be set as

$$s\phi = 0\text{: } \cos^2 i_{c0} = \frac{3}{7} \qquad (i_{c0} = 49.1^\circ) \tag{55}$$

$$c\phi = 0\text{: } \cos^2 i_{c0} = \frac{1}{5} \qquad (i_{c0} = 63.4^\circ) \tag{56}$$

where the values in the parentheses are those that are limited within $0 \le i_{c0} \le \pi/2$. These inclinations are the same as the special inclinations [14]. By using Eqs. (52–56), all the secular terms in Δd can be eliminated.

Next, parts of the periodic terms are eliminated by properly setting the initial values of the deputy. Here, the components of Δp_{y0}, Δp_{z0}, Δv_{x0}, and Δv_{z0}, except for Δp_{x0} and Δv_{y0} that are set in Eqs. (52) and (53), respectively, are considered. The term $\bar{\epsilon}$ that does not depend on Δp_{x0} and Δv_{y0} in ϵ of Eq. (51) is defined. $\bar{\epsilon}$ is expressed as

$$\bar{\epsilon} = \mathbf{w}^T\mathbf{C}\overline{\delta\mathbf{p}_0} \tag{57}$$

$$\overline{\delta\mathbf{p}_0} = [\,\Delta p_{y0} \quad \Delta p_{z0} \quad \Delta v_{x0} \quad \Delta v_{z0}\,]^T$$
$$\mathbf{w} = [\,\sin\theta_c \quad \cos\theta_c - 1 \quad \sin 2\theta_c \quad \cos 2\theta_c - 1 \quad 1\,]^T$$

where **C** is the following matrix:

$$\mathbf{C}=\begin{bmatrix}-s\phi & 0 & \frac{2s\phi}{n_{c0}} & 0\\ c\phi & 0 & -\frac{2c\phi}{n_{c0}} & 0\\ 0 & \frac{\sqrt{3}c\phi}{4} & -\frac{3s\phi}{4n_{c0}} & \frac{\sqrt{3}s\phi}{4n_{c0}}\\ 0 & \frac{\sqrt{3}s\phi}{4} & \frac{3c\phi}{4n_{c0}} & -\frac{\sqrt{3}c\phi}{4n_{c0}}\\ c\phi & \frac{\sqrt{3}}{2}s\phi & 0 & 0\end{bmatrix} \tag{58}$$

The rank of matrix **C** becomes different in the cases of $s\phi=0$ and $c\phi=0$. Hereafter, these two cases are considered separately.

a) Case $s\phi=0$

In this case, $\mathrm{rank}(\mathbf{C})=4$. Although the term proportional to $\sin\theta_c$ in Δd_c cannot be eliminated, other terms that are proportional to $\cos\theta_c$, $\sin 2\theta_c$, and $\cos 2\theta_c$ and the bias term at $\theta_c=0$ can be eliminated. Then, each component of $\overline{\delta\mathbf{p}_0}$ is obtained as

$$\Delta p_{y0}=0 \tag{59}$$

$$\Delta p_{z0}=\frac{2\alpha d_c\sin 2i_{c0}}{c\phi} \tag{60}$$

$$\Delta v_{x0}=-\frac{31d_c n_{c0}}{64c\phi}[\alpha(1+2\cos^2 i_{c0})-2e_{c0}\cos\omega_{c0}] \tag{61}$$

$$\Delta v_{z0}=\frac{\sqrt{3}d_c n_{c0}}{192c\phi}[\alpha(-61+406\cos^2 i_{c0})+90e_{c0}\cos\omega_{c0}] \tag{62}$$

When the initial values are set according to Eqs. (52), (53), and (59–62), and the inclination of the chief is set as $\cos i_{c0}=\sqrt{3/7}$ and $\sin i_{c0}=\sqrt{4/7}$ so as to satisfy Eq. (55), Δd in this case becomes

$$\begin{aligned}\Delta d=\frac{d_c}{224}[&15\alpha\cos 4\theta_c-(13\alpha-14e_{c0}\cos\omega_{c0})\cos 3\theta_c\\ &+2(36\alpha+7e_{c0}\sin\omega_{c0})\sin 3\theta_c\\ &-2(12\alpha+77e_{c0}\sin\omega_{c0})\sin\theta_c-2(\alpha+7e_{c0}\cos\omega_{c0})]\end{aligned} \tag{63}$$

b) Case $c\phi=0$

In this case, $\mathrm{rank}(\mathbf{C})=3$. Here, we eliminate the terms that are proportional to $\sin\theta_c$ and $\sin 2\theta_c$, and the bias term at $\theta_c=0$, by suitably setting Δp_{z0}, Δv_{x0}, and Δv_{z0}. Each component of $\overline{\delta\mathbf{p}_0}$ is expressed as

$$\Delta p_{y0}=0 \tag{64}$$

$$\Delta p_{z0}=0 \tag{65}$$

$$\Delta v_{x0}=-\frac{d_c n_{c0}}{32s\phi}(13\sqrt{3}\alpha\sin 2i_{c0}-9e_{c0}\sin\omega_{c0}) \tag{66}$$

$$\Delta v_{z0}=-\frac{d_c n_{c0}}{32s\phi}(23\alpha\sin 2i_{c0}+7\sqrt{3}e_{c0}\sin\omega_{c0}) \tag{67}$$

When the initial values are set according to Eqs. (52), (53), and (64–67), and the inclination of the chief is set as $\cos i_{c0}=\sqrt{1/5}$ and $\sin i_{c0}=\sqrt{4/5}$ so as to satisfy Eq. (56), Δd in this case becomes

$$\begin{aligned}\Delta d=\frac{d_c}{160}[&-15\alpha\cos 4\theta_c+(7\alpha-10e_{c0}\cos\omega_{c0})\cos 3\theta_c\\ &-2(12\sqrt{3}\alpha+5e_{c0}\sin\omega_{c0})\sin 3\theta_c\\ &-2(\alpha-30e_{c0}\cos\omega_{c0})\cos 2\theta_c\\ &+11(7\alpha-10e_{c0}\cos\omega_{c0})\cos\theta_c-(67\alpha-60e_{c0}\cos\omega_{c0})]\end{aligned} \tag{68}$$

IV. Numerical Examples

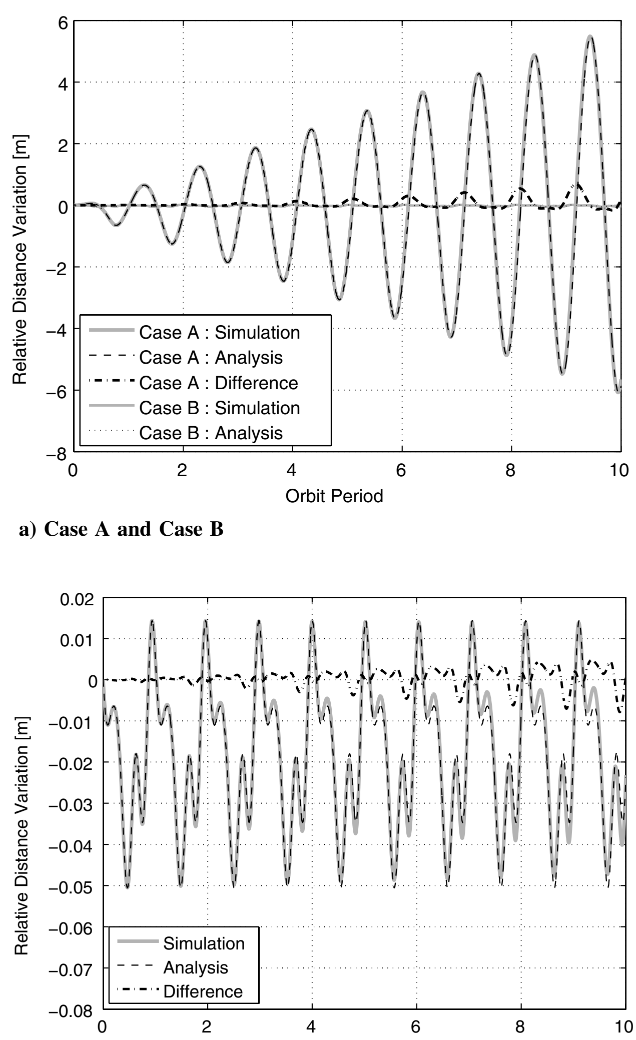
We calculate the variation in the relative distance caused by J_2, where the deputy makes a GCO, and compare the simulation results with the analytical ones. We independently carry out simulations of the chief and the deputy under the effects of J_2 and calculate the variation in the relative distance. The equations of motion used in these simulations are

$$\frac{\mathrm{d}^2\mathbf{r}_c}{\mathrm{d}t^2}=-\frac{\mu}{r_c^3}\mathbf{r}_c+\mathbf{f}_c \tag{69}$$

$$\frac{\mathrm{d}^2\mathbf{r}_d}{\mathrm{d}t^2}=-\frac{\mu}{r_d^3}\mathbf{r}_d+\mathbf{f}_d \tag{70}$$

where $\mathrm{d}^2\mathbf{r}_c/\mathrm{d}t^2$ and $\mathrm{d}^2\mathbf{r}_d/\mathrm{d}t^2$ are the second time derivatives of the position vector of the chief and the deputy, and $\mathbf{f}_c$ and $\mathbf{f}_d$ are the J_2 forces per unit mass exerted on the chief and the deputy, respectively. $\mathbf{f}_c$ and $\mathbf{f}_d$ are expressed as

$$\mathbf{f}_c=\frac{3}{2}\frac{\mu J_2R^2}{r_c^7}[(5r_{ck}^2-r_c^2)\mathbf{r}_c-2r_c^2r_{ck}\hat{\mathbf{k}}] \tag{71}$$



a) Case A and Case B

b) Case B (magnified)

Fig. 1 Relative distance variation in GCO ($\phi=90°$, $i_{c0}=63.4°$).

$$\mathbf{f}_d = \frac{3}{2}\frac{\mu J_2 R^2}{r_d^7}[(5r_{dk}^2 - r_d^2)\mathbf{r}_d - 2r_d^2 r_{dk}\hat{\mathbf{k}}] \tag{72}$$

where $\hat{\mathbf{k}}$ is the unit vector directed from the center of the Earth to the North Pole, and r_{ck} and r_{dk} are the components of $\mathbf{r}_c$ and $\mathbf{r}_d$, respectively, in the direction of $\hat{\mathbf{k}}$ such that

$$r_{ck} = \mathbf{r}_c \cdot \hat{\mathbf{k}}, \qquad r_{dk} = \mathbf{r}_d \cdot \hat{\mathbf{k}} \tag{73}$$

By expressing the components of $\mathbf{f}_c$ in the Hill coordinates, Eqs. (1–3) are derived [16].

In this simulation, the orbit elements of the chief at the ascending node ($\theta_c = 0$) are set as

$$a_{c0} = 7178 \text{ [km]}, \qquad \Omega_{c0} = 0°, \qquad e_{c0} = 0.0005$$
$$\omega_{c0} = 90° \qquad i_{c0} = 63.4°$$

The relative distance between the chief and deputy at the ascending node d_c is 50 m. The initial phase angle ϕ of the deputy at the chief ascending node is set as $\phi = \pi/2$ so as to satisfy Eq. (56). The initial values of the deputy are given in the following two cases: Case A—$\delta\mathbf{p}_0$ [Eq. (47)], and Case B—$\delta\mathbf{p}_0$ [Eq. (47)] $+\Delta\delta\mathbf{p}_0$, where Eqs. (52), (53), and (64–67) are used for $\Delta\delta\mathbf{p}_0$.

Simulation studies of the GCO are performed by using Eqs. (69–72) for 10 orbit periods of the chief. Figure 1a shows Δd in Eq. (49), Δd_c given in the appendix, and the difference between the two ($\Delta d - \Delta d_c$) for Case A; and Δd in Eqs. (49) and (68) for Case B. As shown in this figure, secular terms exist in Δd because of e_{c0}, and these terms gradually increase. The simulation results agree well with the analytical ones. Figure 1b shows the magnified figure of Δd in Eqs. (49) and (68) and the difference between the two for Case B. As shown in this figure, it is clear that the secular terms are eliminated, and the distance variation is largely decreased in Case B; the simulation results also agree well with the analytical ones. The methods to obtain the suitable initial values by the use of the mean orbit elements [13,17] are difficult to apply to this case, because the transformation matrix between the mean orbit elements and the osculating orbit elements has singularity at this inclination [9].

V. Conclusions

In this study, the effects of the J_2 perturbation caused by the Earth's gravitational potential on the relative distance between two spacecraft within one orbit period were analyzed by focusing on a case wherein the chief spacecraft is in a near-circular orbit and a deputy spacecraft makes a formation with a constant distance from the chief. We used the osculating orbit elements of the chief and obtained a state transition matrix of the relative position and velocity of the deputy. By using this state transition matrix, the relative distance variation was expressed analytically, and the initial values of the deputy were obtained without using the mean orbit elements in order to decrease the variation. From the results of the analysis and numerical simulations, it was found that the distance variation for the deputy with respect to the chief was largely reduced by properly setting the aforementioned values. The results of this study can be applied to a telescope mission, for example, in cases where a large focal distance is achieved by the formation flying of two spacecraft and where the relative distance between the spacecraft is required to be precisely maintained. In particular, the osculating orbit element descriptions of the relative distance variation and the proper initial values of the deputy are expressed in simple forms and, thus, are easily applied to control problems of the relative distance.

Appendix

By substituting $\delta\mathbf{p}_0$ from Eq. (47) into $\delta\mathbf{p}(0)$ and calculating Δd using Eq. (49), we obtain the following Δd_c for a GCO where the second- and higher-order terms of J_2 and e_{c0} are neglected:

$$\begin{aligned}
\Delta d_c = \alpha d_c \Bigg[& \frac{9}{16}(-1 + 3\cos^2 i_{c0})\theta_c \sin(2\theta_c + 2\phi) \\
& - \frac{9}{32}\sin^2 i_{c0}\theta_c(\sin 2\theta_c + \sin 2\phi) - \frac{3}{4}(2\sqrt{3}\cos\phi \sin 2i_{c0} \\
& + 3\sin\phi)\theta_c \cos(\theta_c + \phi) + \frac{15}{128}\sin^2 i_{c0}\cos(4\theta_c + 2\phi) \\
& - \frac{1}{32}(1 + 2\cos^2 i_{c0})\cos(3\theta_c + 2\phi) + \frac{3\sqrt{3}}{16}\sin 2i_{c0}\sin(3\theta_c + 2\phi) \\
& + \frac{21}{32}(3\cos^2 i_{c0} - 1)\cos(2\theta_c + 2\phi) + \frac{1}{32}(11\cos^2 i_{c0} \\
& + 25)\cos 2\theta_c + \frac{\sqrt{3}}{16}\sin 2i_{c0}[3\sin(2\theta_c + 2\phi) + \sin 2\theta_c \\
& - 3\sin(\theta_c + 2\phi) + 10\sin\theta_c] - \frac{3}{32}(11 + 14\cos^2 i_{c0})\cos(\theta_c \\
& + 2\phi) + \frac{1}{16}(1 - 10\cos^2 i_{c0})\cos\theta_c - \frac{3\sqrt{3}}{16}\sin 2i_{c0}\sin 2\phi \\
& + \frac{1}{128}(205 - 61\cos^2 i_{c0})\cos 2\phi - \frac{9}{32}(3 - \cos^2 i_{c0})\Bigg] \\
& + e_{c0} d_c \Bigg[-\frac{3}{2}(\sin\omega_{c0}\cos\phi + 3\cos\omega_{c0}\sin\phi)\theta_c \cos(\theta_c + \phi) \\
& + \frac{1}{16}\cos(3\theta_c - \omega_{c0} + 2\phi) - \frac{3}{32}[7\cos(2\theta_c - \omega_{c0} + 2\phi) \\
& - 3\cos(2\theta_c + \omega_{c0}) - 18\sin(\omega_{c0} + \phi)\sin(2\theta_c + \phi)] \\
& + \frac{1}{16}[3\cos(\theta_c + \omega_{c0} + 2\phi) + 6\cos(\theta_c - \omega_{c0} + 2\phi) \\
& + 14\cos(\theta_c + \omega_{c0}) + 8\cos(\theta_c - \omega_{c0})] \\
& + \frac{1}{32}[21\cos(\omega_{c0} + 2\phi) + 7\cos(\omega_{c0} - 2\phi) - 80\cos\omega_{c0}] \Bigg]
\end{aligned}$$

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